Random walks and random permutations

2001 J. Phys. A: Math. Gen. 34 L417

(http://iopscience.iop.org/0305-4470/34/31/101)

J. Phys. A: Math. Gen. **34** (2001) L417–L423 PII: S0305-4470(01)20528-9

LETTER TO THE EDITOR

Random walks and random permutations

P J Forrester

Department of Mathematics and Statistics, University of Melbourne, Parkville 3052, Australia

Received 26 March 2001
Published 27 July 2001
Online at stacks.iop.org/JPhysA/34/L417

Abstract
A connection is made between the random-turns model of vicious walkers and random permutations indexed by their increasing subsequences. Consequently the scaled distribution of the maximum displacements in a particular asymmeteric version of the model can be determined to be the same as the scaled distribution of the eigenvalues at the soft edge of the GUE (random Hermitian matrices). The scaling of the distribution gives the maximum mean displacement μ after t time steps as $\mu = (2t)^{1/2}$ with standard deviation proportional to $\mu^{1/3}$. The exponent $1/3$ is typical of a large class of two-dimensional growth problems.

PACS numbers: 05.40.Fb, 05.50.+q, 64.60.-i, 02.10.Ab

Non-intersecting (vicious) random walkers were introduced into statistical mechanics [6] as models of domain walls and wetting in two-dimensional lattice systems, and have also received attention as exactly solvable systems [3, 4, 7–9]. They can be viewed as directed lattice paths which start at sites say on the x-axis and finish on sites on the line $y = n$, with the additional constraint that the paths do not touch or overlap. Alternatively, vicious random walkers can be described as the stochastic evolution of particles on a one-dimensional lattice, which at each tick of the clock move to the right or to the left with a certain probability, subject to the constraint that no two particles can occupy the one lattice site.

Our interest is in the random-turns vicious-walker model [6, 9]. Here, in the stochastic evolution picture, at each time step t ($t = 1, 2, \ldots$) one particle is selected at random and moved one lattice site to the right with probability w_1, or one lattice site to the left with probability w_{-1} ($w_{-1} + w_1 = 1$). However, if the lattice site to the right (left) is already occupied, then the chosen walker moves to the left (right) with a probability of one unless this lattice site is also occupied. In the latter situation another walker is selected at random and the procedure repeated until one walker has been moved. The move of precisely one walker thus determines the state at time interval t. An example of some typical configurations in the directed paths picture is given in figure 1. We remark that the random-turns vicious-walker model can also be regarded as a particular asymmetric-exclusion process [17].

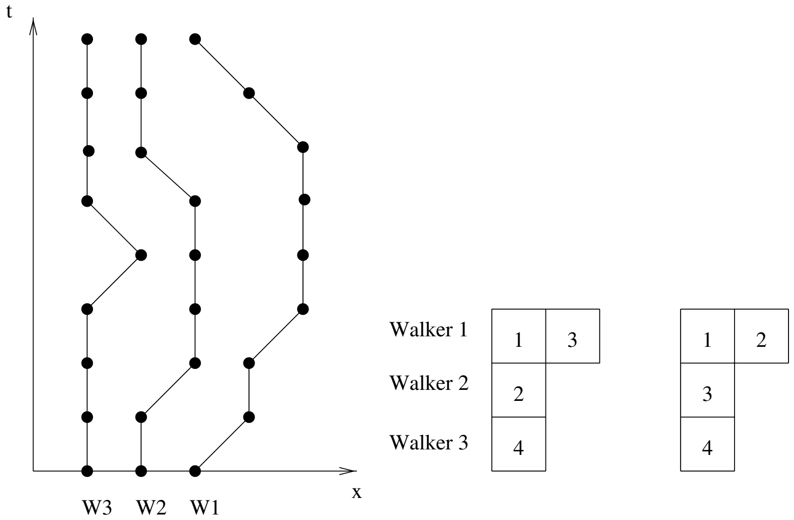

Figure 1. A particular configuration of three random-turns walkers, labelled W1, W2, W3 from right to left. performing eight steps in the sequence R^4L^4. The walk can conveniently be represented diagrammatically as shown on the right according to the rules specified in the text.

Two aspects of the theory of the random-turns vicious-walker model are the subject of this letter. The first concerns the number of configurations that have p walkers initially on adjacent lattice sites ($l = 1, \ldots, p$) on the one-dimensional line, and have the walkers returned to the same sites after $2n$ time steps (for an odd number of time steps this is not possible and thus the reason for $2n$). This will be shown to be simply related to the number of permutations of $\{1, 2, \ldots, n\}$ such that the maximum increasing subsequence has length no greater than p. Then existing results concerning the distribution of the maximum increasing subsequence of a random permutation will be used to determine the distribution of the maximum displacement of the walkers for a variant of the random-turns model. In this variant the number of walkers is greater than or equal to n, and all walkers must move to the right for time steps up to n, while they must move to the left thereafter, returning to their starting points at time $2n$.

To count the configurations we need to label the particles by their initial location on the one-dimensional line $l = 1, \ldots, p$. At each time step one walker will move one lattice site to the right (R) or left (L), subject to the rule that no two particles can occupy the same lattice site (in the counting problem we take $w_1 = w_{-1}$). The constraint that the particles return to their initial positions after $2n$ steps requires that for each walker the number of steps to the right equals the number of steps to the left after $2n$ steps, and that in total there are n steps R and n steps L. This latter fact allows the walks to be partitioned according to the ordering of the R and L steps, of which there are $\binom{2n}{n}$ possibilities. The simplest of these is R^nL^n, which means the first n steps are all to the right, while the final n steps are all to the left. We now pose the question: for a given combination of $\{R^n, L^n\}$ corresponding to a particular ordering of R and L steps, how many distinct configurations of the p random-turns walkers are there?

Consider the particular ordering R^nL^n. Let the rightmost walker be referred to as walker 1 and the second rightmost walker be referred to as walker 2 etc. We will represent each configuration diagrammatically as a pair of tableau (the technical definition of a tableau is given below) corresponding to the steps R^n and L^n, respectively. In each tableau the jth row corresponds to the jth walker. For the steps R^n, each time walker j moves a square is placed in row j immediately to the right of any other squares, or in the first column or row j if the walker moves for the first time. In that square is recorded the number of the step at which the walker moves. For the steps L^n a second tableau is constructed following the same procedure, except that we start with the last step (which is now regarded as step 1) and work backwards. An example of such a diagrammatical representation is given in figure 1.

In general this procedure will give a pair of diagrams in which the rows are weakly decreasing in length (following from the non-intersection condition). By construction each of the n squares must be labelled by a different integer $1, \dots, n$ (referred to as the content) with the numbers strictly increasing along the rows and down the columns. These requirements regarding the shape of the diagram together with the requirements regarding the content specify a standard tableau (see for example, [11]). Notice too that each tableau in the pair must have the same shape since the walkers must return. Conversely, given a pair of standard tableaux of the same shape, each with content $\{1, \dots, n\}$, according to the above specifications we can write down a unique walker configuration in the sequence $R^n L^n$. Thus there is a bijection between pairs of standard tableaux and walker paths in the sequence $R^n L^n$. We remark that this is not the first time a bijection between Young tableaux and vicious-walker paths has been observed: in [12] a bijection between semi-standard tableaux and the configurations in a special case of the lock-step random-walker model [6, 8] was identified.

In the bijective correspondence the pairs of standard tableaux are constrained so that the number of rows is less than or equal to p (the number of walkers), or equivalently that the length of the first column is less than or equal to p. But such pairs of standard tableaux are well known (see for example, [11]) to be in bijective correspondence, in this case the Robinson–Schensted correspondence, with permutations of $\{1, \dots, n\}$ such that the length of each increasing subsequence is less than or equal to p. Hence we have enumerated the number of walks in terms of such permutations.

Proposition 1. *Consider p random-turns walkers, initially equally spaced one unit apart and returning to their initial position after $2n$ steps. Suppose the walkers make their steps in the sequence $R^n L^n$. The total number of distinct configurations equals the number of permutations of $\{1, \dots, n\}$ such that the length of the maximum increasing subsequence is less than or equal to p.*

Consider now another sequence of n R and n L steps. This sequence can be transformed into the sequence $R^n L^n$ by elementary transpositions s_i which interchange the ith and $(i+1)$th members of the sequence, assumed to be L and R, respectively. Likewise, we can define the corresponding action of s_i on the lattice paths and so obtain a bijection between distinct configurations with walks following the sequence $R^n L^n$, and distinct configurations with walks following some combination of the sequence $R^n L^n$. We assume step i is opposite in direction to step $i + 1$, and defined the action of s_i to interchange the order these two steps are made. Thus if originally walker k moves to the left at step i and walker k' moves to the right at step $i + 1$, then after the action of s_i walker k' moves to the right at step i and walker k moves to the left at step $i + 1$. The corresponding action on the lattice paths is depicted in figure 2.

If the same walker originally moved to the left at step i then to the right at step $i + 1$, it may happen that the resulting lattice path after interchange is inadmissible, in that the new (local) configuration intersects with an existing path (note that this cannot happen in the first two cases of figure 2). In such a circumstance we move the configuration to the right in the last diagram of figure 2 until a permissible configuration is obtained (after so moving the left–right pairs two vertical lines, corresponding to a stationary walker, take its place). This is illustrated in figure 3. Note that in all cases $s_i^2 = 1$ and thus the procedure is invertible. Furthermore, the braid relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ are satisfied so the correspondence is independent of the

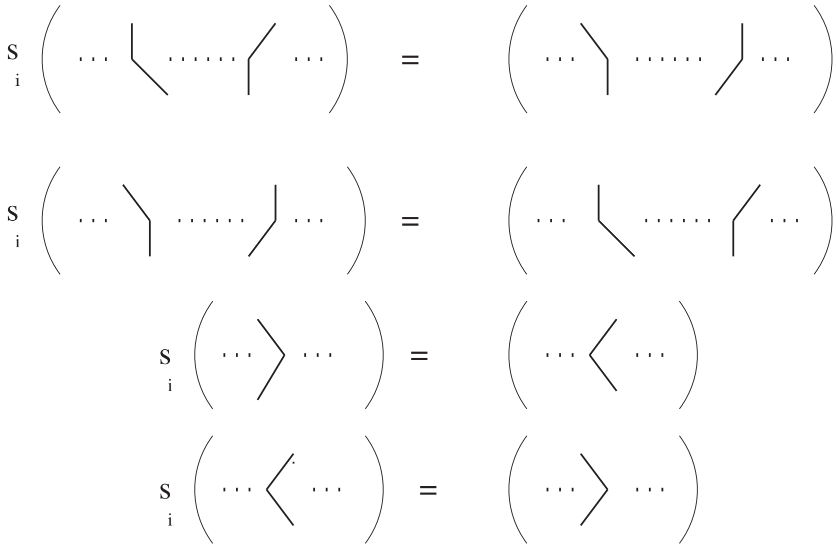

Figure 2. In the first two cases distinct walkers move at steps i and $i + 1$, while in the last two cases the same walker moves, and the resulting configuration is assumed to be admissible.

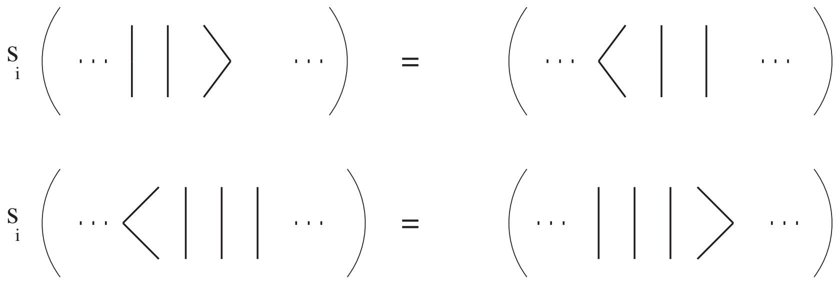

Figure 3. Here the same walker moves at steps i and $i + 1$. Since the rule of figure 2 would lead to inadmissible configurations in each case, s_i acts by propagating the new configuration to the left and right in the two cases, respectively, until an admissible configuration is obtained.

order of application of the transpositions.

The action of the elementary transpositions thus described give a bijection between lattice paths with steps in the sequence $R^n L^n$, and lattice paths with steps in a sequence of any particular combination of $\{R^n, L^n\}$. Thus by making use of proposition 1 we can solve the enumeration problem for all such walks.

Proposition 2. *The result of proposition 1 for the number of walks in the sequence $R^n L^n$ also applies for any combination of $\{R^n, L^n\}$.*

It is of interest to note that there are multiple integral formulae for both the number of random permutations of $\{1, \ldots, n\}$ with the length of the maximum increasing subsequence less than or equal to p, and the total number of random-turns paths with p walkers starting at sites $l'_1, \ldots, l'_p$ and finishing at sites $l_1, \ldots, l_p$ in $2n$ steps. Let us denote these numbers by f_{np} and $Z_{2n}(l'_1, \ldots, l'_p; l_1, \ldots, l_p)$, respectively. Then we have [19]

$$f_{np} = \frac{(n!)^2}{(2n)!} \frac{1}{p!} \frac{1}{(2\pi)^p} \int_{-\pi}^{\pi} \mathrm{d}\theta_1 \cdots \int_{-\pi}^{\pi} \mathrm{d}\theta_p \left(\sum_{j=1}^{p} 2\cos\theta_j \right)^{2n} \prod_{1 \leqslant \alpha < \beta \leqslant p} |\mathrm{e}^{\mathrm{i}\theta_\alpha} - \mathrm{e}^{\mathrm{i}\theta_\beta}|^2 \tag{1}$$

and [9]

$$Z_{2n}(l'_1,\ldots,l'_p;l_1,\ldots,l_p)$$
$$= \frac{1}{(2\pi)^p}\int_{-\pi}^{\pi}\mathrm{d}\theta_1\cdots\int_{-\pi}^{\pi}\mathrm{d}\theta_p\left(\sum_{j=1}^{p}2\cos\theta_j\right)^{2n}\det[\mathrm{e}^{-\mathrm{i}(l_\alpha-l'_\beta)\theta_\alpha}]_{\alpha,\beta=1,\ldots,p}. \tag{2}$$

Let us adapt (2) to propositions 1 and 2 by choosing $l_j = l'_j = j$ $(j = 1,\ldots,p)$. Now

$$\det[\mathrm{e}^{-\mathrm{i}(\alpha-\beta)\theta_\alpha}]_{\alpha,\beta=1,\ldots,p} = \prod_{j=1}^{p}\mathrm{e}^{-\mathrm{i}(j-1)\theta_j}\det[\mathrm{e}^{\mathrm{i}(\beta-1)\theta_\alpha}]_{\alpha,\beta=1,\ldots,p}$$
$$= \prod_{j=1}^{p}\mathrm{e}^{-\mathrm{i}(j-1)\theta_j}\prod_{1\leqslant\alpha<\beta\leqslant p}(\mathrm{e}^{\mathrm{i}\theta_\beta}-\mathrm{e}^{\mathrm{i}\theta_\alpha})$$

where the final equality follows from the Vandermonde formula. Substituting this into (2) shows that the integrand consists of a symmetric factor, the non-symmetric factor $\prod_{j=1}^{p}\mathrm{e}^{-(j-1)\theta_j}$, and an antisymmetric factor. Now of course the value of the integral is unchanged if we symmetrize the integrand and divide by $p!$. Since the final factor is antisymmetric, symmetrizing the integrand is equivalent to antisymmetrizing the non-symmetric factor $\prod_{j=1}^{p}\mathrm{e}^{-(j-1)\theta_j}$. This gives another Vandermonde product and so we have

$$Z_{2n}(\{l'_j = j\}_{j=1,\ldots,p};\{l_k = k\}_{k=1,\ldots,p})$$
$$= \frac{1}{p!}\frac{1}{(2\pi)^p}\int_{-\pi}^{\pi}\mathrm{d}\theta_1\cdots\int_{-\pi}^{\pi}\mathrm{d}\theta_p\left(\sum_{j=1}^{p}2\cos\theta_j\right)^{2n}\prod_{1\leqslant\alpha<\beta\leqslant p}|\mathrm{e}^{\mathrm{i}\theta_\beta}-\mathrm{e}^{\mathrm{i}\theta_\alpha}|^2. \tag{3}$$

Comparing (1) and (3) gives

$$Z_{2n}(\{l'_j = j\}_{j=1,\ldots,p};\{l_k = k\}_{k=1,\ldots,p}) = \binom{2n}{n}f_{np} \tag{4}$$

which is of course also an immediate corollary of propositions 1 and 2. However, once having deduced (4), the formula (1) for f_{np} follows as a special case of (2).

We know from the derivation of proposition 1 that there is a bijection between configurations of p random-turns walkers performing $2n$ steps in the sequence R^nL^n before returning to their initial positions of all one unit apart, and pairs of standard tableaux each of the same shape consisting of n boxes and with no more than p rows. Furthermore, in the bijection the length of row j corresponds to the maximum displacement of walker j to the right of its starting point (this occurs at step n). Thus if we choose $p \geqslant n$ the constraint on the number of rows is empty as the $p-n$ leftmost walkers never get a chance to move, and the bijection is with pairs of tableaux of the same shape with n boxes each. In such a situation the asymptotics of the row lengths are known precisely [1, 2, 14, 18]. We can therefore interpret these results in the random-walker setting.

Proposition 3. *Denote by l_j the displacement at time step n of walker j from its initial position p–j. Define the scaled displacements by*

$$\tilde{l}_j := n^{1/3}\left(\frac{l_j}{n^{1/2}}-2\right) \tag{5}$$

and the corresponding scaled k-point distribution function by

$$\rho_k(\tilde{l}_1,\ldots,\tilde{l}_k) := \lim_{n\to\infty}\left(\frac{1}{n^{1/6}}\right)^k\rho_k^{(n)}(\tilde{l}_1,\ldots,\tilde{l}_k)$$

where $\rho_k^{(n)}$ *denotes the k-point distribution for the walker problem in the finite system. Then from the results of [2, 14, 18] for the tableau problem we have*

$$\rho_k(\tilde{l}_1, \ldots, \tilde{l}_k) = \det[K(\tilde{l}_\alpha, \tilde{l}_\beta)]_{\alpha,\beta=1,\ldots,k} \tag{6}$$

where

$$K(x, y) := \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x - y}$$

with $\mathrm{Ai}(x)$ *denoting the Airy function.*

Distribution (6) is precisely the scaled distribution of the eigenvalues at the edge of the spectrum for GUE (random Hermitian matrices) [10, 20]. One recalls that a matrix X from the GUE is specified by having a joint distribution of elements proportional to the Gaussian $\exp(-X^2)$, and the largest eigenvalue occurs in the neighbourhood of $\lambda = \sqrt{2N}$, which is referred to as the soft edge. By making the scaling $\lambda \mapsto \sqrt{2N} + \lambda/\sqrt{2}N^{1/6}$ the corresponding distribution function has a well defined scaled limit which is given by (6). Perhaps more relevantly to the random-walker problem, (6) coincides with the scaled distribution for free fermions on a line confined by a one-body harmonic potential, at the edge of the support of the density. The relevance is that there is a well known relationship between continuous models of non-intersecting walkers and free fermions (see for example, [5]).

Regarding some physical features of proposition 3, note from (5) that the average displacement is $\mu = 2n^{1/2}$ (for a recent independent proof of this result see [13]), with standard deviation proportional to $(4n)^{1/6} = \mu^\chi$, $\chi = 1/3$. As emphasized in [15], the exponent $\chi = 1/3$ is typical of two-dimensional growth models (it can be derived from the one-dimensional Burgers equation describing such processes [21]). On this point we recall that vicious-walker paths fixed at the endpoints as in proposition 3 form the well known (see for example, [16]) terrace–step–kink model of a crystal surface.

The financial support of the ARC, including funds to support the visit of G Olshanski whose lectures benefitted the present work, are acknowledged. The author is grateful for the considered comments of a referee. Also, the remarks of T H Baker on the original manuscript are appreciated.

References

[1] Baik J, Dieft P and Johansson K On the distribution of the length of the longest increasing subsequence of random permutations. *J. Am. Math. Soc.* **12** 1119–78

[2] Borodin A, Okounkov A and Olshanski G 2000 Asymptotics of Plancherel measures for symmetric groups *J. Am. Math. Soc.* **13** 481–515

[3] Brak R and Owczarek A L 1999 A combinatorial interpetation of the free-fermion condition of the six vertex model *J. Phys. A: Math. Gen.* **32** 3497–503

[4] Brak R and Owczarek A L 1999 Exact solution of n-directed non-intersecting walks interacting with one or two boundaries *J. Phys. A: Math. Gen.* **32** 2921–9

[5] den Nijs M 1988 *Phase Transitions and Critical Phenomena* vol 12, ed C Domb and J L Lebowitz p 219 (Orlando: Springer)

[6] Fisher M E 1984 Walks, walls, wetting and melting *J. Stat. Phys.* **34** 667–729

[7] Forrester P J 1989 Probability of survival for vicious walkers near a cliff *J. Phys. A: Math. Gen.* **22** L609–13

[8] Forrester P J 1990 Exact solution of the lockstep model of vicious walkers *J. Phys. A: Math. Gen.* **23** 1259–73

[9] Forrester P J 1991 Exact results for vicious walker models of domain walls *J. Phys. A: Math. Gen.* **24** 203–18

[10] Forrester P J 1993 The spectrum edge of random matrix ensembles *Nucl. Phys.* B **402** 709–28

[11] Fulton W 1997 Young Tableaux *London Mathematical Society Student Texts* (Cambridge: Cambridge University Press)

[12] Guttmann A J, Owczarek A L and Viennot X G 1998 Vicious walkers and Young tableaux I: without walls. *J. Phys. A: Math. Gen.* **31** 8123–35

[13] Johansson K 1998 The longest increasing subsequence in a random permutation and a unitary random matrix model *Math. Res. Lett.* **5** 63–82

[14] Johansson K 1999 Discrete orthogonal polynomial ensembles and the Plancherel measure *Preprint* math.CO/9906120

[15] Johansson K 2000 Shape fluctuations and random matrices *Commun. Math. Phys.* **209** 437–76

[16] Joós B, Einstein T L and Bartelt N C 1991 Distribution of terrace widths on a vicinal surface within the one-dimensional free-fermion model *Phys. Rev.* B **43** 8153–62

[17] Liggert T M 1985 *Interacting Particle Systems* (New York: Springer)

[18] Okounkov A 2000 Random matrices and random permutations *Int. Math. Res. Not.* **20** 1043–95
(Okounkov A 1999 *Preprint* math.CO/9903176)

[19] Rains E M 1998 Increasing subsequences and the classical groups *Elect. J. Combin.* **5** R12

[20] Tracy C A and Widom H 1994 Level-spacing distributions and the Airy kernel *Commun. Math. Phys.* **159** 151–74

[21] van Beijeren H 1991 Fluctuations in the motions of mass and of patterns in one-dimensional driven diffusion equations *J. Stat. Phys.* **63** 47–57